W13. Let r_a, r_b and r_c be the length of the radii of excircles of a triangle $\triangle ABC$ with circumradius R and inradius r. Let a, b and c be the length of the sides of $\triangle ABC$. Prove that:

$$\frac{a^2}{r_b r_c} + \frac{b^2}{r_c r_a} + \frac{c^2}{r_a r_b} \geq 4.$$

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Solution by Arkady Alt, **San Jose**, **California**, **USA**.

Let s be semiperimeter of $\triangle ABC$. Since $r_a = \frac{sr}{s-a}, r_b = \frac{sr}{s-b}, r_c = \frac{sr}{s-c}$

then $\sum_{cyc} \frac{a^2}{r_b r_c} = \frac{1}{s^2 r^2} \sum_{cyc} a^2(s-b)(s-c) = \frac{1}{s^2 r^2} \sum_{cyc} a^2(s-b)(s-c) =$

$\frac{1}{s^2 r^2} \sum (a^2 s^2 - s(a^2 b + a^2 c) + a^2 bc) = \frac{1}{s^2 r^2} (s^2(a^2+b^2+c^2) - 2s^2(ab+bc+ca) + 5sabc).$

Noting that $abc = 4Rrs, ab+bc+ca = s^2 + 4Rr + r^2$ and $R \geq 2r$ (Euler's Inequality)

we obtain $\sum_{cyc} \frac{a^2}{r_b r_c} = \frac{1}{s^2 r^2} (s^2(a^2+b^2+c^2) - 2s^2(ab+bc+ca) + 20Rrs^2) =$

$\frac{1}{r^2} ((a^2+b^2+c^2) - 2(ab+bc+ca) + 20Rr) = \frac{1}{r^2} (4s^2 - 4(ab+bc+ca) + 20Rr) =$

$\frac{1}{r^2} (4s^2 - 4(s^2 + 4Rr + r^2) + 20Rr) = \frac{4}{r}(R - r) \geq \frac{4}{r}(2r - r) = 4.$